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Non-Ergodicity in a 1-D Particle Process with Variable Length

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We present a 1-D random particle process with uniform local interaction, which displays some form of non-ergodicity, similar to contact processes, but more unexpected. Particles, enumerated by integer numbers, interact at every step of the discrete time only with their nearest neighbors. Every particle has two possible states, called minus and plus. At every time step two transformations occur. The first one turns every minus into plus with probability β independently from what happens at other places and thereby favors pluses against minuses. The second one is "impartial." Under its action, whenever a plus is a left neighbor of a minus, both disappear with probability α independently from presence and fate of other pairs of this sort. If β is small enough by comparison with α^2 and we start with "all minuses," the minuses never die out.

KEY WORDS: Particle random process; one-dimensional local interaction; contact processes; non-ergodicity; Peierls contour estimation; variable length; duality of planar graphs.

1. INTRODUCTION AND DECLARATION OF THEOREMS

The bulk of modern studies of interacting particle systems is based on the assumption that the set of sites, also called the space, does not change in the process of interaction. Elements of this space, also called components, may be in different states, e.g., 0 and 1, often interpreted as absence vs. presence of a particle, and may go from one state to another, which may be interpreted as birth or death of a particle, but the sites themselves do not appear or disappear in the process of functioning. Operators and processes, which do not create or eliminate sites, will be called *constant-length* ones. The main purpose of ref. 8 was to introduce a new class of one-dimensional

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particle processes with continuous time, called *variable-length* processes. In the present article we consider one process of this sort with discrete time.

We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{Z}$ the set of integer numbers and $\mathbb{Z}_+$ the set of natural numbers (including zero). We choose a non-empty finite or countable set $\mathscr{A}$, called *alphabet*, and call its elements *letters*. We call a *word* in the alphabet $\mathscr{A}$ any finite sequence of terms, everyone of which is an element of $\mathscr{A}$. The length of a word is the number of letters in it. Any letter may be treated as a word of length one. There is the empty word, whose length is zero. Let us call the *dictionary* and denote by $dict(\mathscr{A})$ the set of words in the alphabet $\mathscr{A}$. We assume that comma and brackets do not belong to $\mathscr{A}$ and if we write several words and letters one after another, perhaps, separated by commas and included in brackets, they form one word (commas and brackets eliminated), which we call their *concatenation*.

We consider a configuration space $\mathscr{A}^{\mathbb{Z}}$, the set of bi-infinite sequences, whose terms are elements of $\mathscr{A}$. We call a *thin cylinder* any set of the form

$$\{s \in \mathscr{A}^{\mathbb{Z}} : s_i = a_i \text{ for all } i \in [m, n]\}, \tag{1}$$

where s_i are components of s, i.e., variables, whose values are elements of $\mathscr{A}$, and $a_i \in \mathscr{A}$ are constants.

We consider probability measures, that is normed measures on $\mathscr{A}^{\mathbb{Z}}$, that is on the σ-algebra generated by thin cylinders. In the usual way we define translations on $\mathbb{Z}$, then on $\mathscr{A}^{\mathbb{Z}}$, then on the set of normed measures on $\mathscr{A}^{\mathbb{Z}}$ and call a measure *uniform* if it is invariant under all translations. We shall concentrate our attention on uniform measures. Dealing with uniform measures, we may use the following simplified notation for any word $W = (a_1,..., a_n)$:

$$\mu(W) = \mu(a_1,..., a_n) = \mu(s_{i+1} = a_1,..., s_{i+n} = a_n). \tag{2}$$

Since μ is uniform, the probability (2) does not depend on i and we call it the *frequency* of the word W in the measure μ. Any uniform measure on $\mathscr{A}^{\mathbb{Z}}$ is determined by its values (2) on all words in the alphabet $\mathscr{A}$. To form a uniform measure, the numbers $\mu(W)$ must be non-negative and for any word W (including the empty one)

$$\mu(W) = \sum_{a \in \mathscr{A}} \mu(W, a) = \sum_{a \in \mathscr{A}} \mu(a, W), \tag{3}$$

where (W, a) and (a, W) are concatenations of the word W and letter a in the two possible orders. As is well-known, a measure can serve as a probability distribution if it is normed, that is its value on all the space is 1.

A uniform measure is normed if its value on the empty word is 1. We denote by $\mathcal{M}_{\mathcal{A}}$ the set of uniform measures on $\mathcal{A}^{\mathbb{Z}}$. By convergence in $\mathcal{M}_{\mathcal{A}}$ we mean convergence on all words in the alphabet $\mathcal{A}$.

As usual, we write events and functions after measures. For example, μf means the mean of the function f according to the measure μ, $\mu(E)$ means the measure μ of the event E, which is the same as $\mu I(E)$, where $I(E)$ is the indicator function of E. When operators are involved, they are written between measures and events or functions. For example, μPQ means the measure obtained from measure μ by application of operator P first and application of operator Q second and $\mu PQ(E)$ means the value of this measure on the event E.

As usual, an operator P acting on $\mathcal{M}_{\mathcal{A}}$ is called *linear* if for any $a, b \in \mathbb{R}$ and any $\mu, \nu \in \mathcal{M}_{\mathcal{A}}$

$$(a \cdot \mu + b \cdot \nu)\, P = a \cdot (\mu P) + b \cdot (\nu P).$$

Operators with constant length, considered traditionally, typically were linear. We shall see that even a simple variable-length operator may be non-linear.

Our main results pertain to the case, when the alphabet $\mathcal{A}$ has only two elements, which we denote by $\ominus$ and $\oplus$ and call *minus* and *plus*. In this case our operators act on $\mathcal{M}_{\{\ominus, \oplus\}}$, the set of normed uniform measures on configuration space $\{\ominus, \oplus\}^{\mathbb{Z}}$. Let us define two operators acting on $\mathcal{M}_{\{\ominus, \oplus\}}$ depending on parameters α, β, where β takes all values in $[0, 1]$, but $0 < \alpha < 1$, because the case $\alpha = 0$ is trivial and the case $\alpha = 1$ is troublesome.

The operator, which we call *flip* and denote by Flip_β, is well-known. It is constant-length and linear. Under its action any minus turns into plus with probability β independently of others. We need to represent our operators using independent auxiliary variables. Let us define Flip_β, denoting by $x_i \in \{\ominus, \oplus\}$ for all $i \in \mathbb{Z}$ the coordinates of that space $\{\ominus, \oplus\}^{\mathbb{Z}}$, where the initial measure μ is given. Also, we use mutually independent variables F_i for all $i \in \mathbb{Z}$, each taking two values called *move* and *stay*, distributed according to a product-measure π, defined as follows:

$$F_i = \begin{cases} \textit{move} & \text{with probability} \quad \beta, \\ \textit{stay} & \text{with probability} \quad 1-\beta. \end{cases}$$

Finally, we have a third set of variables $y_i \in \{\ominus, \oplus\}$ for all $i \in \mathbb{Z}$, on which the measure $\mu\, \text{Flip}_\beta$ is induced by the product of the measures μ and π with the map

$$y_i = \begin{cases} \ominus & \text{if} \quad x_i = \ominus \quad \text{and} \quad F_i = \textit{stay}, \\ \oplus & \text{in all the other cases.} \end{cases} \tag{4}$$

Clearly, operator Flip_β can be applied to any $\mu \in \mathscr{M}_{\{\ominus, \oplus\}}$ and produces a measure in $\mathscr{M}_{\{\ominus, \oplus\}}$, preserving the set of uniform normed measures.

The *Annihilation* operator $\text{Ann}_\alpha\colon \mathscr{M}_{\{\ominus, \oplus\}} \to \mathscr{M}_{\{\ominus, \oplus\}}$ is variable-length and seems to have never been mentioned except ref. 8. It is because of this operator we decided to consider only uniform measures. Informally speaking, whenever a word $(\oplus, \ominus)$ occurs in the infinite configuration, it disappears with probability α independently of all the other occurences. Notice that under the action of Ann_α the eliminated sites disappear completely rather than go to another state. If we considered a similar process on finite configurations, every act of elimination of a word $(\oplus, \ominus)$ would decrease the length of the configuration by two. We shall define Ann_α as a superposition of two operators: $\text{Ann}_\alpha = \text{Duel}_\alpha \text{ Clean}$ (first Duel_α, then Clean). You may imagine that when Duel_α is applied, a duel occurs between every pair of $\oplus$ and $\ominus$ occupying ith and $(i+1)$th sites respectively (in this order only). If the command *fire!* is given, which occurs for every such pair independently with a probability α, the duellists kill each other. Otherwise the command *stop!* is given and nothing happens. When Clean is applied, the dead bodies are disposed of and the live sites close ranks.

Now let us define operator Duel_α, a linear constant-length operator transforming any measure on $\{\ominus, \oplus\}^{\mathbb{Z}}$ into a measure on $\{\ominus, \oplus, \odot\}^{\mathbb{Z}}$, where $\odot$ is a third state introduced especially for this occasion and called *dead*. States different from dead, that is minus and plus, are called *live*. Let us call $x_i \in \{\ominus, \oplus\}, i \in \mathbb{Z}$ the coordinates of the space $\{\ominus, \oplus\}^{\mathbb{Z}}$, where the original measure μ is defined. Also, we use mutually independent variables A_i for all $i \in \mathbb{Z}$, each taking two values called *fire* and *stop*, distributed according to a product-measure π, defined as follows:

$$A_i = \begin{cases} \textit{fire} & \text{with probability} \quad \alpha, \\ \textit{stop} & \text{with probability} \quad 1-\alpha \end{cases} \tag{5}$$

for any $i \in \mathbb{Z}$ independently of all the other components and of the measure μ. We denote by $y_i \in \{\ominus, \oplus, \odot\}$ the coordinates of the space, where the measure $\mu\,\text{Duel}_\alpha$ is induced by the product of μ and π, with the following map:

$$y_i = \begin{cases} \odot & \text{if} \quad x_i = \oplus, \quad x_{i+1} = \ominus \quad \text{and} \quad A_{i+1} = \textit{fire}, \\ \odot & \text{if} \quad x_{i-1} = \oplus, \quad x_i = \ominus \quad \text{and} \quad A_i = \textit{fire}, \\ x_i & \text{in all the other cases.} \end{cases}$$

Also notice that $\mu(\oplus, \ominus) \leqslant 1/2$ for any $\mu \in \mathcal{M}_{\{\ominus, \oplus\}}$, because $\mu(\oplus, \ominus) = \mu(\ominus, \oplus)$ and their sum does not exceed 1. Hence, since $\alpha < 1$ and from the definition of Duel_α

$$\mu \, \text{Duel}_\alpha(\odot) = 2\alpha \cdot \mu(\oplus, \ominus) < 1. \tag{6}$$

Now let us define a variable-length operator Clean: $\mathcal{M}_{\{\ominus, \oplus, \odot\}} \to \mathcal{M}_{\{\ominus, \oplus\}}$. For any $\mu \in \mathcal{M}_{\{\ominus, \oplus, \odot\}}$ we directly express the values of μ Clean on all words in the alphabet $\{\ominus, \oplus\}$ in terms of the values of μ on all words in the alphabet $\{\ominus, \oplus, \odot\}$. By definition, we set μ Clean on the empty word to be 1. For any non-empty word $W = (a_0,..., a_k) \in dict(\ominus, \oplus)$ we define $\mu \, \text{Clean}(W)$ as follows:

$$\mu \, \text{Clean}(a_0,..., a_k)$$
$$= \frac{1}{1 - \mu(\odot)} \sum_{n_1,..., n_k = 0}^{\infty} \mu(a_0 \odot^{n_1} a_1 \odot^{n_2} a_2 \cdots \odot^{n_{k-1}} a_{k-1} \odot^{n_k} a_k), \tag{7}$$

where $\odot^n$ means the word consisting of n letters, everyone of which is $\odot$ (in fact, the empty word if $n = 0$). So

$$a_0 \odot^{n_1} a_1 \odot^{n_2} a_2 \cdots \odot^{n_{k-1}} a_{k-1} \odot^{n_k} a_k$$

means the word, which starts with letter a_0, then go n_1 letters $\odot$ (in fact, none if $n_1 = 0$), then goes letter a_1, then n_2 letters $\odot$, then letter a_2, and so on till n_{k-1} letters $\odot$, then letter a_{k-1}, then n_k letters $\odot$, and finally letter a_k and the summing is done over all $n_1,..., n_k$ from zero to infinity. Notice that the formula (7) is non-linear, whence the well-developed theory of linear operators cannot be applied here, which adds to the difficulty of dealing with variable-length processes. Notice also that in the case $k = 0$ the formula (7) turns into

$$\mu \, \text{Clean}(\ominus) = \frac{\mu(\ominus)}{1 - \mu(\odot)}, \qquad \mu \, \text{Clean}(\oplus) = \frac{\mu(\oplus)}{1 - \mu(\odot)}. \tag{8}$$

Is easy to prove that for any $\alpha < 1$ the operator $\text{Ann}_\alpha = \text{Duel}_\alpha \, \text{Clean}$ can be applied to any measure in $\mathcal{M}_{\{\ominus, \oplus\}}$ and turns it into a measure in $\mathcal{M}_{\{\ominus, \oplus\}}$. Now let us declare our theorems. Let us denote by $\delta_\ominus$ and $\delta_\oplus$ the degenerate measures concentrated in the configurations "all minuses" and "all pluses" respectively. For all natural t we denote

$$\mu_t = \delta_\ominus \, (\text{Flip}_\beta \, \text{Ann}_\alpha)^t. \tag{9}$$

Theorem 1. For all natural t the frequency of pluses in the measure μ_t does not exceed $300 \cdot \beta/\alpha^2$.

Since $\delta_\oplus$ is invariant for $\text{Flip}_\beta \, \text{Ann}_\alpha$ with any α and β, Theorem 1 implies that the operator $\text{Flip}_\beta \, \text{Ann}_\alpha$ cannot be ergodic whenever $\beta < \alpha^2/300$ because in this case μ_t cannot tend to $\delta_\oplus$.

Theorem 2. If $2\beta > \alpha$, the measures μ_t tend to $\delta_\oplus$ when $t \to \infty$.

Taken together, Theorems 1 and 2 show that the sequence of measures μ_t has at least two different modes of behavior. In one mode ($\beta > \alpha/2$) these measures tend to $\delta_\oplus$ when $t \to \infty$ and in the other mode ($\beta < \alpha^2/300$) they do not tend to $\delta_\oplus$.

Theorem 3. Take any $\mu \in \mathscr{M}_{\{\ominus, \oplus\}}$ and suppose that $\beta > 0$ and $(1-\beta) \cdot \mu(\ominus) \leqslant 1/2$. Then the measures $\mu(\text{Flip}_\beta \, \text{Ann}_\alpha)^t$ tend to $\delta_\oplus$ when $t \to \infty$.

Let us denote by $s(\alpha, \beta)$ the supremum of density of $\oplus$ in measure μ_t for all natural t.

Theorem 4. For every $\alpha \in (0, 1)$, $s(\alpha, \beta)$ is not continuous as a function of β.

Our theorems show similarity and difference between our process and the well-known contact processes (see, e.g., refs. 3 and 4). Since our time is discrete, it is better to compare our process with the well-known Stavskaya process, a discrete-time version of contact processes. (See ref. 5 or Example 1.2 on pp. 8–10 of refs. 1 or 6 or Section 6.2 on p. 139 in ref. 7.) Using our notations, Stavskaya process is a sequence of measures $\delta_\ominus (\text{Flip}_\beta \, \text{Stav})^t$, where the deterministic constant-length operator Stav: $\{\ominus, \oplus\}^{\mathbb{Z}} \to \{\ominus, \oplus\}^{\mathbb{Z}}$ is defined by the rule

$$\forall s \in \{\ominus, \oplus\}^{\mathbb{Z}}, \qquad k \in \mathbb{Z}: (s \, \text{Stav})_k = \begin{cases} \oplus & \text{if} \quad s_k = s_{k+1} = \oplus, \\ \ominus & \text{otherwise.} \end{cases}$$

The operator Stav favors minuses against pluses, because it turns any plus into a minus whenever its right neighbor is minus, but never turns minuses into pluses. The operator Flip_β, on the contrary, turns minuses into pluses with a rate β. It is natural that their composition behaves in different ways for large vs. small β, namely, when β is large, minuses die out and when β is small, they do not. Contact processes behave in a similar way. In our case behavior is more unexpected: Flip_β favors pluses for any $\beta > 0$, annihilation is "impartial," but still minuses survive for β/α^2 small enough. Of course, "impartiality" of annihilation should be taken with a tongue in the cheek. In fact, it favors that state, which already prevails—see our Lemma 1.

Unlike our process, as shown by Theorem 3, Stavskaya process does not tend to $\delta_\oplus$ from initial measures in which minuses and pluses are mixed at random in any proportion, provided initial density of minuses is positive and β is small enough. What about our function $s(\alpha, \beta)$, for some situations (contact processes, percolation) its analogs have been proved to be continuous. Theorem 4 shows that our process is different. In this respect (lack of continuity) its behavior may be compared with a first order phase transition. In a forthcoming paper, based on this one, we hope to prove an analog of Theorem 1 for a more symmetric process, in which pluses and minuses turn into each other independently with one and the same rate β.

2. PROOF OF THEOREMS 2, 3, AND 4.

Lemma 1. For any $\mu \in \mathscr{M}_{\{\ominus, \oplus\}}$, if $\mu(\ominus) \leqslant 1/2$, then $\mu\,\mathrm{Ann}_\alpha(\ominus) \leqslant \mu(\ominus)$.

Proof. From the definition of Duel_α

$$\mu\,\mathrm{Duel}_\alpha(\ominus) = \mu(\ominus) - \alpha \cdot \mu(\oplus, \ominus).$$

Hence, from (6) and from the definition of Clean

$$\mu\,\mathrm{Ann}_\alpha(\ominus) = \frac{\mu(\ominus) - \alpha \cdot \mu(\oplus, \ominus)}{1 - 2\alpha \cdot \mu(\oplus, \ominus)},$$

where the denominator is positive since $\alpha < 1$. Now, assuming that $\mu(\ominus) \leqslant 1/2$,

$$\mu(\ominus) - \mu\,\mathrm{Ann}_\alpha(\ominus) = \mu(\ominus) - \frac{\mu(\oplus) - \alpha \cdot \mu(\oplus, \ominus)}{1 - 2\alpha \cdot \mu(\oplus, \ominus)}$$

$$= \frac{\alpha \cdot \mu(\oplus, \ominus)(1 - 2\mu(\ominus))}{1 - 2\alpha \cdot \mu(\oplus, \ominus)} \geqslant 0.$$

Lemma 1 is proved.

Proof of Theorem 3. Since $\mu\,\mathrm{Flip}_\beta(\ominus) = (1-\beta) \cdot \mu(\ominus)$ and $(1-\beta) \cdot \mu(\ominus) \leqslant 1/2$, the frequency of minuses in $\mu\,\mathrm{Flip}_\beta$ does not exceed $1/2$.

Then, from Lemma 1, the frequency of minuses in $\mu\,\mathrm{Flip}_\beta\,\mathrm{Ann}_\alpha$ also does not exceed 1/2. Arguing in this way, we can prove by induction that the frequency of minuses in $\mu(\mathrm{Flip}_\beta\,\mathrm{Ann}_\alpha)^t$ does not exceed $(1-\beta)^{t-1}/2$ for all $t \geqslant 1$, and therefore tends to zero when $t \to \infty$, whence the measure tends to $\delta_\oplus$. Theorem 3 is proved.

Now let us prove Theorem 2. Here the case $\beta = 0$ is impossible and the case $\beta = 1$ is trivial, so let $0 < \beta < 1$. If there is t such that $(1-\beta)\cdot\mu_t(\ominus) \leqslant 1/2$, Theorem 2 follows from Theorem 3. It remains to examine the case when $(1-\beta)\cdot\mu_t(\ominus) > 1/2$ for all t. We shall prove that this case is impossible. Notice that

$$\mu_{t+1}(\ominus) = \frac{(1-\beta)\cdot\mu_t(\ominus) - \alpha\cdot p}{1-2\alpha\cdot p}, \tag{10}$$

where we have denoted $p = \mu_t\,\mathrm{Flip}_\beta(\oplus, \ominus)$. It is easy to prove that the expression (10) is a growing function of p under our conditions. Since $\mu(\oplus, \ominus) \leqslant 1/2$, whence $\alpha\cdot\mu(\oplus, \ominus) \leqslant \alpha/2$, this implies that

$$\mu_{t+1}(\ominus) \leqslant \frac{(1-\beta)\cdot x - \alpha/2}{1-\alpha},$$

where $x = \mu_t(\ominus)$. Therefore

$$\mu_{t+1}(\ominus) - \mu_t(\ominus) \leqslant -\frac{(\beta-\alpha)\,x + \alpha/2}{1-\alpha}.$$

Here the right side is a linear function of x, which equals $-\alpha/(2-2\alpha)$ at $x = 1/(2-2\beta)$ and $-(\beta-\alpha/2)/(1-\alpha)$ at $x = 1$. Both of these values are negative, so

$$\mu_{t+1}(\ominus) - \mu_t(\ominus) \leqslant m,$$

where m is a negative constant, whence $\mu_t(\ominus)$ tends to $-\infty$ when $t \to \infty$, which is impossible, because a probability cannot be negative. Theorem 2 is proved.

Now let us assume that Theorem 1 is also proved and argue towards Theorem 4. Notice that $s(\alpha, \beta)$ cannot take values in $(1/2, 1)$, because if it does, then there is t such that $\mu_t(\oplus) > 1/2$. But then, due to Theorem 3, $\mu_t(\oplus)$ tends to 1 when $t \to \infty$, whence $s(\alpha, \beta) = 1$. Thus for any $\alpha \in (0, 1)$ $s(\alpha, \beta)$ equals 1 if $\beta > \alpha/2$ due to Theorem 2, tends to 0 when $\beta \to 0$ due to Theorem 1 and cannot take values in $(1/2, 1)$ due to Theorem 3, so it cannot be continuous. *Theorem 4 is reduced to Theorem 1*. The remaining part of the article is proof of Theorem 1. Starting now we assume that $\beta < \alpha^2/300$ because otherwise Theorem 1 is obvious.

3. PROCESS v AND ITS GRAPHICAL REPRESENTATION.

Our proof of Theorem 1 is based on two well-known ideas: Peierls' contour method and duality of planar graphs. Both ideas were used, for example, in ref. 6 and if a reader finds it difficult to follow our arguments, he or she may first look at that paper. Also you may look at the figure placed in the Appendix at the end of this article, which illustrates ideas presented below. Let us introduce a process v, which differs from our original process in the following respect. It is not necessary to clean the dead particles out at every time step. We may leave them where they are, but in this case we have to sacrifice locality, namely we must organize interaction of live particles as if dead particles were removed. Starting now, we denote by $x \in \mathbb{Z}$ the space coordinate. We shall also use a natural parameter y, which equals zero at the beginning and increases by one after every application of Flip_β or Ann_α. Thus y increases by two when t in the formula (9) increases by one. Accordingly, we denote by $F(x, t)$ and $A(x, t)$ and call *basic variables* those variables F_i and A_i, which participate in the $(t+1)$th application of $\text{Flip}_\beta\ \text{Ann}_\alpha$. Thus our basic space is

$$\Omega = (\{move, stay\} \times \{fire, stop\})^{\mathbb{Z} \cdot \mathbb{Z}_+}$$

with coordinates

$$(F(x, t)), A(x, t)), \qquad \text{where} \qquad x \in \mathbb{Z}, \quad t \in \mathbb{Z}_+ \tag{11}$$

and with a product measure π, according to which for all x, t

$$\begin{aligned} F(x,t) &= \begin{cases} move & \text{with probability} \quad \beta, \\ stay & \text{with probability} \quad 1-\beta, \end{cases} \\ A(x,t) &= \begin{cases} fire & \text{with probability} \quad \alpha, \\ stop & \text{with probability} \quad 1-\alpha. \end{cases} \end{aligned} \tag{12}$$

Let us denote

$$V = \{(x, y), x \in \mathbb{Z}, y \in \mathbb{Z}_+\}.$$

The sets of pairs $(x, y) \in V$ with a given y are called *y-levels* or just levels. Every pair $(x, y) \in V$ has a state denoted by *state*(x, y), which equals $\ominus$, $\oplus$ or $\odot$ and all their states are functions of $\omega \in \Omega$ defined in the following inductive way.

Base of Induction. $state(x, 0) = \ominus$ for all $x \in \mathbb{Z}$.

Induction step when y is even, say $y = 2t$, where $t \in \mathbb{Z}_+$ (imitating the action of Flip_β). For all $x \in \mathbb{Z}$:

$$state(x, 2t+1) = \begin{cases} \oplus & \text{if} \quad state(x, 2t) = \ominus \text{ and } F(x, t) = move, \\ state(x, 2t) & \text{in all the other cases.} \end{cases}$$

Induction step when y is odd, say $y = 2t+1$, where $t \in \mathbb{Z}_+$ (imitating the action of Ann_α, but without locality). For all $x \in \mathbb{Z}$:

$$state(x, 2t+2) = \begin{cases} \odot & \text{if } state(x, 2t+1) = \ominus \text{ and } A(x, t) = fire \\ & \text{and there is } x' < x \text{ such that } state(x', 2t+1) = \oplus \\ & \text{and } \forall x'' \in \mathbb{Z}\colon x' < x'' < x \Rightarrow state(x'', 2t+1) = \odot; \\ \odot & \text{if } state(x, 2t+1) = \oplus \text{ and there is } x' > x \text{ such that} \\ & state(x', 2t+1) = \ominus \text{ and } A(x', t) = fire \\ & \text{and } \forall x'' \in \mathbb{Z}\colon x < x'' < x' \Rightarrow state(x'', 2t+1) = \odot; \\ state(x, 2t+1) & \text{in all the other cases.} \end{cases}$$

Informally speaking, in this process our particles never disappear and keep the same integer indices, which they had at the beginning. If a particle annihilates, it goes to the dead state $\odot$ and remains in this state forever. Live particles interact as if dead components did not exist. Thus we have an inductionally defined map from Ω to $\{\ominus, \oplus, \odot\}^V$. We denote by ν the measure on $\{\ominus, \oplus, \odot\}^V$ induced by the distribution π of the basic variables (12) with this map and ν_y the distribution of states on the yth level. The process ν is useful for us because

$$\nu_{2t}\,\text{Clean} = \mu_t \tag{13}$$

for all t, which is easy to prove by induction. Also it is easy to prove that

$$\begin{cases} \text{(a)} & \nu(state(x, y) = \ominus) > 0 \text{ for all } (x, y) \in V. \\ \text{(b)} & \text{For any integer } x_0 \text{ and any natural } y \\ & \qquad \nu(\forall x \geqslant x_0 : state(x, y) \neq \ominus) \\ & \qquad\qquad = \nu(\forall x \leqslant x_0 : state(x, y) \neq \ominus) = 0. \\ \text{(c)} & \mu_t(\ominus) > 0 \text{ for all natural } t. \\ \text{(d)} & \text{For any integer } x_0 \text{ and any natural } t \\ & \qquad \mu_t(\forall x \geqslant x_0 : s_x \neq \ominus) = \mu_t(\forall x \leqslant x_0 : s_x \neq \ominus) = 0. \end{cases} \tag{14}$$

Now we go to a graphical representation of the process v. In the following text we shall ignore some events, whose probability is zero. So, reading it, you should mentally insert "almost," "almost all," or "almost sure" whenever necessary. For any $\omega \in \Omega$ we define a graph G. The figure in the Appendix illustrates part of such a graph. Along with describing the graph G, we shall describe how to draw it in a plane, representing vertices by points and edges by curves (in fact, straight segments). The set of vertices of G is

$$V_G = \{(x, y) \in V, \mathit{state}(x, y) \neq \odot\}$$

and every vertex (x, y) is placed at the point (x, y) of the plane, where x and y are the usual orthogonal coordinates, the axis x is horizontal and the axis y is vertical. Graph G has two kinds of edges, which we call *vertical* and *horizontal*. Let us describe them.

Vertical Edges. Any two vertices $(x, y_1), (x, y_2)$ of G, where $y_2 - y_1 = 1$, are connected with a vertical edge. Direction of this edge from (x, y_1) to (x, y_2) is called *north*, the other direction is called *south*. We call (x, y_1) the south neighbor of (x, y_2) and (x, y_2) the north neighbor of (x, y_1).

Horizontal Edges. Any two vertices $(x_1, y), (x_2, y)$ of G, where $x_1 < x_2$, are connected with a horizontal edge if

$$\forall x \in \mathbb{Z}\colon x_1 < x < x_2 \Rightarrow \mathit{state}(x, y) = \odot.$$

Direction of this edge from (x_1, y) to (x_2, y) is called *east*, the other direction is called *west*. We call (x_1, y) the west neighbor of (x_2, y) and (x_2, y) the east neighbor of (x_1, y). Thus G, which has only those edges, which are specified above, is defined. Its edges are represented by straight segments connecting the points representing ends of the edge.

A vertex of G, whose level y is even, always has exactly one west neighbor, exactly one east neighbor and exactly one north neighbor. Also it has exactly one south neighbor, except the case $y = 0$, when it has no south neighbor. A vertex of G, whose level is odd, always has exactly one west neighbor, exactly one east neighbor and exactly one south neighbor. Also it has at most one north neighbor. Due to the definition of G, every vertex of it is in a state $\oplus$ or $\ominus$; in the former case we call it a $\oplus$-vertex, in the latter a $\ominus$-vertex.

It is evident that different edges G do not intersect except common ends. We shall call the *picture* of G its representation in the plane just described. This picture cuts the plane into parts, which we call *faces*. We assume that all the faces are closed. We call two faces neighbors if they have a common edge. Our picture of G has exactly one unbounded face, namely the bottom half of the plane. All the other faces of G are bounded and we call them *boxes*. Every box has the form of a rectangle, sanwiched between two parallel lines at levels y_1 and y_1+1, where y_1 is natural, so it may be denoted

$$\{(x, y) \in \mathbb{R}^2 : x_1 \leqslant x \leqslant x_2, y_1 \leqslant y \leqslant y_1+1\}. \tag{15}$$

For every natural y_1 the boxes sandwiched between the parallel lines at the levels y_1 and y_1+1 form a bi-infinite sequence in which every two next terms have a common side and which we call a *horizontal corridor* at sub-(y_1+1) level. Any box has at least four vertices placed at its corners and has no more vertices on its west, east and north walls, so it has exactly one west neighbor, one east neighbor and one north neighbor. If y_1 is even, the box (15) has no more vertices at its south wall, whence it has exactly one south neighbor. If y_1 is odd, this box (15) has $2k+1$ south neighbors, where k is the number of annihilations, which occured at the $(y_1+1)/2$th application of the operator Ann_α between sites x_1 and x_2.

Like in ref. 6, we use the well-known duality of pictures of graphs. (See a detailed treatment in ref. 2.) Let us describe a graph, which we denote by $\bar{G}$, and its picture, which will be dual of the picture of G. We place that vertex of $\bar{G}$, which is dual of the box (15), at the point

$$\left(\frac{x_1+x_2}{2}, y_1+1-\varepsilon\right), \tag{16}$$

where $\varepsilon > 0$ is chosen for different boxes fifferently, but should be small enough in every case; how small, we shall explain. We shall say that the vertex (16) has a sub-(y_1+1) level. We say that it has a *sub-even* level if y_1+1 is even and has a *sub-odd* level if y_1+1 is odd. There is just one subtlety: that vertex of $\bar{G}$, which is dual of the only unbounded face of the picture of G, is placed "infinitely far" in the negative direction of the axis y and the edges leading to it are rays with the same direction. All the other edges of $\bar{G}$ are straight segments connecting the points representing their ends. Thus the graph $\bar{G}$ and its picture are defined. It is easy to see that for any box the corresponding ε can be chosen so small that the usual conditions of dual pictures be fulfilled.

We shall call *horizontal* those edges of $\bar{G}$, which are dual of vertical edges of G and *vertical* those edges of $\bar{G}$, which are dual of horizontal edges of G. Notice that horizontal edges of $\bar{G}$ are approximately horizontal because values of ε for all vertices of $\bar{G}$ are approximatedly equal to zero. For any natural y the vertices of $\bar{G}$, which are at sub-$(y+1)$ level, and horizontal edges, connecting them, form a bi-infinite path, which we call a *horizontal path* at sub-$(y+1)$ level and which is dual of the sub-$(y+1)$ corridor. Any bounded face of $\bar{G}$ is sandwiched between horizontal paths at the levels sub-y and sub-$(y+1)$. Unbounded faces of $\bar{G}$ are dual of vertices of G at the level zero. They are unbounded half-strips, which fill all the halfplane below the horizontal path at the sub-1 level. A face of $\bar{G}$ is called a west (respectively east, north or south) neighbor of another face of $\bar{G}$ if their corresponding vertices of G are in the same relation.

According to what we said about vertices of G at even levels, any face of $\bar{G}$ at an even level has exactly one west neighbor, exactly one east neighbor and exactly one north neighbor. Also it has exactly one south neighbor, except the case $y=0$, when it has no south neighbor. Whenever $y>0$, we call these faces of $\bar{G}$ *rectangles*. In fact, all of them approximatedly are rectangles. According to what we said about vertices of G at odd levels, any face of $\bar{G}$ at an odd level has at most one north neighbor. If it has one, we call it a *trapezium*, otherwise we call it a *triangle*. Indeed, these faces approximatedly are trapeziums and triangles.

4. A CHAIN OF EQUALITIES AND INEQUALITIES

Starting now, we fix an arbitrary natural number T. Our overall goal is to estimate $\mu_T(\oplus)$ uniformly in T. Due to (14), $\mu_T(\ominus)$ is positive, so the fraction $\mu_T(\oplus)/\mu_T(\ominus)$ makes sense and it is sufficient to estimate this fraction. To reduce our task further, let us prove that

$$\mu_T(\oplus)=\sum_{k=1}^{\infty}\mu_T(\ominus,\oplus^k). \tag{17}$$

Indeed, let us consider the event of presence of a plus at a certain site and cut it into pieces according to the number of pluses on the left side of this site. From item (d) of (14) this number is finite a.s. whence (17) follows. Then from (17)

$$\mu_T(\oplus)\leqslant\frac{\mu_T(\oplus)}{\mu_T(\ominus)}=\sum_{k=1}^{\infty}\frac{\mu_T(\ominus,\oplus^k)}{\mu_T(\ominus)}. \tag{18}$$

To reduce our task further, we concentrate our attention on Ω_0, the set of those $\omega\in\Omega$, for which $state(0,2T)=\ominus$. For any $\omega\in\Omega_0$ we denote by $x_{\max}(\omega)$ the smallest positive x such that $state(x,2T)=\ominus$. Due to item (b) of (14), $x_{\max}(\omega)$ exists a.s. Let us call *flowers* all those pairs $(x, 2T)$, where $0 < x < x_{\max}(\omega)$, for which $state(x, 2T) = \oplus$. We denote by $\phi(\omega)$ the number of flowers. Since $x_{\max}(\omega)$ exists a.s., $\phi(\omega)$ is finite a.s. For any $k = 1, 2, 3,\ldots$ we denote by Ω_k the set of those $\omega \in \Omega_0$ for which $\phi(\omega) \geqslant k$. Notice that $\Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots$ Let us prove for all k that

$$\frac{\pi(\Omega_k)}{\pi(\Omega_0)} = \frac{\mu_T(\ominus, \oplus^k)}{\mu_T(\ominus)}. \tag{19}$$

Notice that $\pi(\Omega_0) = \nu_{2T}(\ominus)$. But from (13) and (8)

$$\mu_T(\ominus) = \nu_{2T}\ \mathrm{Clean}(\ominus) = \frac{\nu_{2T}(\ominus)}{1 - \nu_{2T}(\odot)},$$

whence

$$\pi(\Omega_0) = \nu_{2T}(\ominus) = \mu_T(\ominus)(1 - \nu_{2T}(\odot)). \tag{20}$$

On the other hand, Ω_k is the set of those $\omega \in \Omega_0$, for which the configuration at the level $2T$ contains one of the words

$$\ominus\ \odot^{n_1}\ \oplus\ \odot^{n_2} \cdots \oplus\ \odot^{n_{k-1}}\ \oplus\ \odot^{n_k}\ \oplus$$

starting at the 0th component. Therefore

$$\pi(\Omega_k) = \sum_{n_1,\ldots,n_k=0}^{\infty} \nu_{2T}(\ominus\ \odot^{n_1} \oplus, \ldots,\ \odot^{n_k} \oplus).$$

But from (13) and (7)

$$\begin{aligned}\mu_T(\ominus, \oplus^k) &= \nu_{2T}\ \mathrm{Clean}(\ominus, \oplus^k)\\ &= \frac{1}{1 - \nu_{2T}(\odot)} \sum_{n_1,\ldots,n_k=0}^{\infty} \nu_{2T}\ (\ominus\ \odot^{n_1}\ \oplus, \ldots,\ \odot^{n_k} \oplus).\end{aligned}$$

Thus

$$\pi(\Omega_k) = \mu_T(\ominus, \oplus^k) \cdot (1 - \nu_{2T}(\odot)).$$

Dividing this by (20), we get (19). Now we can sum (19) over k and use (18) to obtain

$$\frac{\mu_T(\oplus)}{\mu_T(\ominus)} = \sum_{k=1}^{\infty} \frac{\mu_T(\ominus, \oplus^k)}{\mu_T(\ominus)} = \sum_{k=1}^{\infty} \frac{\pi(\Omega_k)}{\pi(\Omega_0)}. \tag{21}$$

Remember an old gardener's wisdom: no flowers without roots. Let us take any $\omega \in \Omega_1$ and call a path in G *north-west* if its every step goes north or west. Let us call a vertex of G a *root* if there is a north-west path from this vertex to some flower, all the vertices of this path having a state $\oplus$. In particular, all the flowers are roots. Vertices of G, which are not roots, are called *non-roots*. The set of roots is finite a.s. for the same reason why the set of flowers is finite a.s., namely because T is fixed and therefore $x_{\max}(\omega)$ exists a.s. Our estimation is based on building a "contour" around all the roots. Let us call a set S of vertices of a graph *connected* in this graph if for any two elements of this set there is a path in this graph connecting them, in which all the vertices belong to S.

Lemma 2. For any $\omega \in \Omega_1$: (a) The set of roots is non-empty, finite and connected in G. (b) The set of non-roots is infinite and connected in G.

Proof. Most of these statements are evident. To prove that the set of non-roots is connected, let us denote by S_{in} the set of those (x, y), for which $0 < x < x_{\max}(\omega)$ and $0 < y \leqslant 2T$. We denote by S_{out} the set of those vertices of G, which do not belong to S_{in}. It is evident that all the roots belong to S_{in} and that S_{out} is connected. It remains to prove that any non-root in S_{in} is connected with S_{out} by a path, which avoids roots. Let us take any non-root $(x_0, y_0) \in S_{\text{in}}$ and define the set S_0 as follows: a vertex (x, y) belongs to S_0 if there is a path in G connecting (x, y) with x_0, y_0, all the vertices of which are non-roots. Let us denote by t_0 the minimal level of elements of S_0. Now let us look for the most western one of those elements of S_0, which are at the level t_0. If it does not exist, then (s_0, t_0) is connected with S_{out} with a path avoiding roots. Let us assume that it exists. If its state is $\oplus$, then it is a root, because its west neighbor is a root. If its state is $\ominus$, then its south neighbor's state also is a minus, hence it is a non-root, which contradicts our choice of t_0. In both cases we get a contradiction. Lemma 2 is proved.

Let us call *dual-roots* those faces of $\bar{G}$, which are dual of roots, and denote by U the union of dual-roots. Since every dual-root is bounded, U is also bounded and closed since we assume all faces to be closed. Hence from Lemma 2, U is homeomorphic to a closed disk (provided the set of roots is finite). Then the boundary of U is a closed curve, which includes the east side of the rectangle dual of the vertex $(0, 2T)$. So this closed curve includes V_0, the north end of this side, and we may assume that it starts and ends at V_0 and surrounds U in the counter-clockwise direction. This curve can be represented as a path in $\bar{G}$, which we denote by $tour(\omega)$ because it is determined by ω. Now we need to classify all the possible forms of $tour(\omega)$. To this end we need to classify all steps which $tour(\omega)$ may include, that is

Table I

Step in G starting at a $\oplus$-vertex	Type	Associated event	Associated variable
Step west at an even level	1	trivial	none
Step west at an odd level	1′	trivial	none
Step from $(x, 2t+1)$ to $(x, 2t)$ if $F(x, t) = move$	2	$F(x, t) = move$	$F(x, t)$
Step from $(x, 2t+1)$ to $(x, 2t)$ if $F(x, t) = stay$	2′	$F(x, t) = stay$	$F(x, t)$
Step south from an even to an odd level	2″	trivial	none
Step from $(x, 2t+1)$ to its east neighbor if $A(x, t) = fire$	3	$A(x, t) = fire$	$A(x, t)$
Step from $(x, 2t+1)$ to its east neighbor if $A(x, t) = stop$	4	$A(x, t) = stop$	$A(x, t)$
Step east at an even level	4′	trivial	none
Step north	5	trivial	none

some steps in $\bar{G}$. We shall start by classifying some steps in the original graph G. Let us call *types* elements of the set

$$\{1, 1', 2, 2', 2'', 3, 4, 4', 5\}. \tag{22}$$

We shall attribute types to those and only those steps in G, which start at $\oplus$-vertices. All the cases, which may occur, are listed in Table I.

Steps, having the word "trivial" in the third column, are called *trivial*, other steps are called *non-trivial*. To every step in G, which has a type, we attribute an *associated event*. For every trivial step the associated event is Ω and is also called trivial. Non-trivial events are represented in Table I by their conditions. For every non-trivial step we also define an *associated basic variable*, which is shown in the last column. Also every step in G, which has a type, has a *chance*. For typographical reasons chances are shown in the next table, but you can easily infer them right now because chance always equals the probability of the associated event. We shall use the same 1-to-1 correspondence between steps in G and steps in $\bar{G}$ as was defined in ref. 6 and in more detail in ref. 2. Here it is:

If an edge $\bar{e}$ of $\bar{G}$ is dual of an edge e of G, then for each direction of e the dual direction of $\bar{e}$ is the direction from **right** to **left** when we go along e in the given direction. (23)

Table II

Step in $\bar{G}$ having a $\oplus$-face on its left side	Type	Chance	Shift
Step south across an even level	1	1	$(0, -1)$
Step south across an odd level	$1'$	1	$(0, -1)$
"move" step east at a sub-odd level	2	β	$(1, 0)$
"stay" step east at a sub-odd level	$2'$	$1-\beta$	$(1, 0)$
Step east at a sub-even level	$2''$	1	$(1, 0)$
"fire" step north across an odd level	3	α	$(-1, 1)$
"stop" step north across an odd level	4	$1-\alpha$	$(0, 1)$
Step north across an even level	$4'$	1	$(0, 1)$
Step west	5	1	$(-1, 0)$

Type, event and chance, attributed to a step in G, are attributed to its dual step in $\bar{G}$ also. Since a step in G has a type if and only if it starts from a $\oplus$-vertex, a step in $\bar{G}$ has a type if and only if it has a $\oplus$-face on its left side.

You may imagine Tables I and II as one table, which is cut into two parts for typographical reasons. The last column of Table II shows *shifts* defined for all types. Shift is a two-dimensional vector, whose components are called *Hshift* and *Vshift* (abbreviations for horizontal shift and vertical shift). The first column of Table II is formally redundant because it follows from what was said in the first column of Table I; however, it helps to understand why shifts are defined in this way. Chances shown in the third column equal probabilities of events shown in the previous table.

Lemma 3. For any $\omega \in \Omega_1$: (a) all the steps of the path $tour(\omega)$ have types and (b) the path $tour(\omega)$ is a concatenation of two paths, which we denote by $bag(\omega)$ and $lid(\omega)$, with the following properties: all the steps of $bag(\omega)$ have types different from 5; $lid(\omega)$ has $\phi(\omega)$ steps, all of which have type 5.

Proof of (a). Since every step in $tour(\omega)$ has a $\oplus$-face on its left side, it has a type.

Proof of (b). Let us call *dual-flowers* those faces of $\bar{G}$, which are dual of flowers. Notice that all of them are rectangles. Let us follow the contour of U in the opposite, that is clockwise, direction starting at V_0. Then our first $\phi(\omega)$ steps will pass north sides of dual-flowers in the east direction. Therefore, the last $\phi(\omega)$ steps of $tour(\omega)$ are north sides of dual-flowers passed in the west direction. Now notice that if a step in $tour(\omega)$ has type 5, it has a dual-flower on its left side. Indeed, if a step of $tour(\omega)$ has type 5, it has a root on its left, that is south, side and a non-root on its right, that is north, side. Since Flip_β cannot turn pluses into minuses, this is possible only if this step has a flower on its left side. Also notice that if some step of $tour(\omega)$ has type 5, then all the steps following it (if any) also have types 5. Indeed, this step has a dual-flower on its left side. But then it is at the sub-$(2t+1)$ level, whence the next step, if any, has to be in the same direction, that is west, and on the same level, that is also at the north side of a dual-flower, so its type is 5. Lemma 3 is proved.

Let us examine $bag(\omega)$. We start by two observations:

$$\left.\begin{array}{ll}\text{(a)} & \text{If } bag(\omega) \text{ includes a step type 2, then this step has a} \\ & \ominus\text{-face on its right (that is, south) side.} \\ \text{(b)} & \text{If } bag(\omega) \text{ includes a step type 3, 4 or } 4'\text{, then this step} \\ & \text{has a } \ominus\text{-face on its right (that is, east) side.}\end{array}\right\} \tag{24}$$

Indeed, in both cases, if there were a $\oplus$-face there, it would be a dual-root, but the contour surrounding U cannot separate dual-roots from each other.

Any sequence of types is called a *code*. By shift of a code we mean the sum of shifts of its terms and by chance of a code we mean the product of chances of its terms. If all the steps of a path p have types, we denote code(p) and call the *code* of p the sequence of types of steps of p. By shift and chance of such a path we mean shift and chance of its code. From Lemma 3, $bag(\omega)$ has a code and we need to study it. Let us call a path in $\bar{G}$ *well-placed* if it starts at V_0, all its steps have types and all the basic variables associated with its steps are independent from each other and from Ω_0. Given some $\omega \in \Omega_0$ and a code C, we say that ω *realizes* C if the graph $\bar{G}$ contains a well-placed path p, such that the code of p equals C.

Lemma 4. Every $\omega \in \Omega_1$ realizes the code of $bag(\omega)$.

Proof. Let us classify all the basic variables $F(x, t)$ and $A(x, t)$, whose parameter t is between zero and T, into two classes: *left basic variables*—those with $x \leqslant 0$ and *right basic variables*—those with $x > 0$. Let us denote by A_{left} the σ-algebra generated by the left basic variables and A_{right} the σ-algebra generated by the right basic variables. Of course, any event in A_{right} is independent from any event in A_{left}. Let us prove that $\Omega_0 \in A_{\text{left}}$. This follows from a more general statement: for any $y \geqslant 0$, any $n \geqslant 0$ and any $a_{-n}, \ldots, a_{-1} \in \{\ominus, \oplus, \odot\}$ the event

$$\begin{aligned} &state(-n, y) = a_{-n}, state(-n+1, y) \\ &\qquad = a_{-n+1}, \ldots, state(-1, y) = a_{-1}, state(0, y) = \ominus \end{aligned}$$

belongs to A_{left}. This statement is easy to prove by induction in y. Now notice that for any $\omega \in \Omega_1$ all the basic variables associated with steps of $tour(\omega)$ belong to A_{right} and are different from each other because the contour of U cannot pass one and the same edge twice. Hence $tour(\omega)$ is well-placed. Lemma 4 is proved.

For any code C we denote by $real(C)$ the set of those $\omega \in \Omega_0$, which realize C. It is easy to prove for any code C by induction in the length of codes that

$$\frac{\pi(real(C))}{\pi(\Omega_0)} \leqslant \text{chance}(C). \tag{25}$$

Hence, due to Lemma 4, for any k

$$\frac{\pi(\Omega_k)}{\pi(\Omega_0)} \leqslant \frac{\sum \pi(real(\text{code}(bag(\omega))))}{\pi(\Omega_0)} \leqslant \sum \text{chance}(bag(\omega)), \tag{26}$$

where both sums are taken over all different code($bag(\omega)$) for $\omega \in \Omega_k$. To estimate the last sum, for every natural k we define a set of codes, which we denote LC_k and whose elements we call *k-legal codes*. A code $C = (c_1, ..., c_n)$ belongs to LC_k if it satisfies the following conditions:

$$\begin{cases} \text{(LC-a)} & c_1 = 1 \text{ and } c_n = 4'. \\ \text{(LC-b)} & \text{All the terms of C belong to the list } 1, 1', 2, 3, 4, 4'. \\ \text{(LC-c)} & \text{All the pairs } (c_i, c_{i+1}) \text{ belong to the list} \\ & 11', 1'1, 1'2, 21, 22, 23, 24, 31', 34', 44', 4'2, 4'3, 4'4. \\ \text{(LC-d)} & \text{Hshift}(C) \geqslant k \text{ and Vshift}(C) = 0. \end{cases} \tag{27}$$

Since $\text{LC}_1 \supseteq \text{LC}_2 \supseteq \text{LC}_3 \supseteq \ldots$, we denote $\text{LC} = \text{LC}_1$ and call elements of LC just *legal codes*. Of course, the definition of legal codes is chosen to fit codes of $bag(\omega)$ as the following shows.

Lemma 5. For all $\omega \in \Omega_k$ the code of $bag(\omega)$ belongs to LC_k.

Proof. let us prove in turn that the code of $bag(\omega)$ satisfies all the conditions of (27).

Proof of Condition (LC-a). The code of $bag(\omega)$ starts with 1 because our path starts with the east side of the face $(0, 2T)$, passing it from north to south, so, according to Table II, its type is 1. The code of $bag(\omega)$ ends with $4'$ because $bag(\omega)$ can make a step to the topmost level from a lower level only by a step type $4'$.

Proof of Condition (LC-b). From Lemma 3, the code of $bag(\omega)$ cannot contain 5. Let us prove that code($bag(\omega)$) cannot contain $2'$ or $2''$. Indeed, suppose that some step in $bag(\omega)$ has type $2'$ or $2''$. Then this step has a root on its left, that is north side. But then its south side also is a root, which is impossible, since steps of $tour(\omega)$ cannot separate roots from each other.

Proof of Condition (LC-c). Let us present several arguments, due to which all the combinations of types (c_i, c_{i+1}), not included in our list, are impossible in the code of $bag(\omega)$.

- Pairs, in which the first term is in the set $\{1, 3, 4\}$ and the second term is in the set $\{1, 2, 3, 4\}$, are impossible, because the first term ends at a sub-even level, but the second term starts at a sub-odd level.
- Pairs, in which the first term is in the set $\{1', 2, 4'\}$ and the second term is in the set $\{1', 4'\}$, are impossible, because the first term ends at a sub-odd level, but the second term starts at a sub-even level.
- Pairs, in which the first term is in the set $\{1, 1'\}$ and the second term is in the set $\{3, 4, 4'\}$, are impossible. To prove this, notice that from every vertex of $\bar{G}$ there goes exactly one step north, so these steps have to follow one and the same edge, but the first term needs a $\oplus$ face on the east side, while the second term needs a $\ominus$ face on the east side due to (24).
- Pairs, in which the first term is 4 and the second term is $1'$, are impossible, because type 4 means that there was no annihilation, so the face on any side of the step having type 4 cannot be a triangle, so the next step cannot go south as $1'$ does.
- Pairs, in which the first term is $4'$ and the second term is 1, are impossible, because they have to follow one and the same edge, but the face on the east side of the step having type $4'$ must have state $\ominus$, while the face on the east side of the step having type 1 must have state $\oplus$.

Proof of Condition (LC-d). Remember the classification of bounded faces of $\bar{G}$ into rectangles, trapeziums and triangles. It is easy to check that the sum of shifts of steps in going around every of these faces is zero. Now let us notice that if two opposite steps in $\bar{G}$ (that is, steps passing one and the same edge in opposite directions) have shifts, the sum of their shifts is zero. Indeed, due to (24), no step opposite of a step type 3 can have a shift because it has a $\ominus$-face on its left side. All the other steps can be classified into four categories, south, east, north and west, so that a step opposite of south is always north, opposite of north is always south, opposite of west is always east and opposite of east is always west and their shifts cancel. Now let us prove that the contour, surrounding U, has shift $(0, 0)$. As we have noticed, the sum of shifts in going around each face included in this union is $(0, 0)$. Hence the sum of all these shifts also is $(0, 0)$. Every side, which does not belong to the contour of the union, belongs to two faces included in it, and is passed in both directions, so these entries cancel, whence $tour(\omega)$ also has shift $(0, 0)$. But from Lemma 3 $tour(\omega)$ is a concatenation of $bag(\omega)$ and $lid(\omega)$, so the sum of their shifts is $(0, 0)$. Since $lid(\omega)$ consists of $\phi(\omega)$ steps type 5, everyone of which has shift $(-1, 0)$, the shift of $lid(\omega)$ is $(-\phi(\omega), 0)$, whence the shift of $bag(\omega)$ is $(\phi(\omega), 0)$. Lemma 5 is proved.

It follows from Lemma 5 and (26) that for all natural k

$$\frac{\pi(\Omega_k)}{\pi(\Omega_0)} \leqslant \sum_{C \in LC_k} \text{chance}(C)). \tag{28}$$

To finish our argument, we need to make a numerical estimation, but it will be too cumbersome to do with so many types. To reduce their number to four, we call *main types* the elements of the set $\{1, 2, 3, 4\}$. All the quantities defined for types are valid for main types. In particular, every main type has a shift and a chance listed in Table II and shown again in Table III. Also every main type has a *rate*, which is shown in the same table:

A *main code* is a finite sequence, all the terms of which are main types. Its *rate* is the product of rates of its terms. For any code C we denote by short(C) the main code obtained from C by deleting all its non-main terms. We shall simplify our task by dealing with short(code($bag(\omega)$)) instead of code($bag(\omega)$). For every natural k we define the set LMC_k, whose elements are called *k-legal main codes*. By definition, a k-legal main code is a main code $C = (c_1,..., c_n)$, which satisfies the following conditions:

$$\begin{cases} \text{(LMC-a)} & c_1 = 1 \\ \text{(LMC-b)} & \text{For every } i = 1,..., n-1 \text{ it is impossible that} \\ & (c_i = 1,\, c_{i+1} = 3) \text{ or } (c_i = 1,\, c_{i+1} = 4) \text{ or } (c_i = 4,\, c_{i+1} = 1). \\ \text{(LMC-c)} & c_n \text{ equals 3 or 4.} \\ \text{(LMC-d)} & \text{Hshift}(C) \geqslant k. \\ \text{(LMC-e)} & \text{Vshift}(C) \geqslant 0. \end{cases} \tag{29}$$

Since $\text{LMC}_1 \supseteq \text{LMC}_2 \supseteq \text{LMC}_3 \supseteq ...$, we denote $\text{LMC} = \text{LMC}_1$ and call elements of LMC just *legal main codes*. You may notice also that from LMC-a), LMC-b) and LMC-c) any legal main code has length at least three, so in fact $\text{LMC} = \text{LMC}_3$, but we shall not use it. For any legal main

Table III

Main type	Shift	Chance	Rate
1	$(0, -1)$	1	1
2	$(1, 0)$	β	2β
3	$(-1, 1)$	α	α
4	$(0, 1)$	$1-\alpha$	$1-\alpha$

code C let us denote by $\text{Long}(C)$ the set of legal codes C' such that $C = \text{short}(C')$. It is easy to observe that if $C' \in \text{Long}(C)$, then C' can be obtained from C by the following procedure:

$$\begin{cases} \text{(a)} & \text{We start with } C. \\ \text{(b)} & \text{After every 1 we insert } 1'. \\ \text{(c)} & \text{After every 4 we insert } 4'. \\ \text{(d)} & \text{Whenever 3 is followed by 1, we insert } 1' \text{ between them.} \\ \text{(e)} & \text{Whenever 3 is followed by 2, we insert either } 1' \text{ or } 4' \text{ between them.} \\ \text{(f)} & \text{Whenever 3 is followed by 3, 4, or 5, we insert } 4' \text{ between them.} \end{cases} \tag{30}$$

Also it is easy to prove that for any main code C and any $C' \in \text{Long}(C)$

$$\text{Hshift}(C') = \text{Hshift}(C), \tag{31}$$

$$\text{Vshift}(C') \leqslant 2 \cdot \text{Vshift}(C). \tag{32}$$

Here (31) is true because C' is obtained from C by inserting only types $1'$ and $4'$, both of which have $\text{Hshift} = 0$. To prove (32), let us classify main types into *horizontal*, namely 2, whose Vshift is zero, and *vertical*, namely all the others. Due to the rule (30) we can establish a 1-to-1 correspondence between vertical terms of C and the terms inserted after them in the course of this procedure. Then Vshift of every newly inserted term is not greater than Vshift of the corresponding vertical term of C. Hence (32) immediately follows.

Lemma 6. For any k, if $C \in LC_k$, then $\text{short}(C) \in LMC_k$.

Proof. Let us assume that $C \in LC_k$ and prove that $\text{short}(C)$ satisfies all the conditions of (29).

Proof of (LMC-a). Is evident.

Proof of (LMC-b). Impossibility of (1, 3). If there is such a combination in short(code($bag(\omega)$)), then there is a combination $(1, 1', 3)$ in code($bag(\omega)$), but combination $(1', 3)$ is impossible there according to (LC-c). Impossibility of (1, 4). If there is such a combination in short(code($bag(\omega)$)), then there is a combination $(1, 1', 4)$ in code($bag(\omega)$), but combination $(1', 4)$ is impossible there according to (LC-c). Impossibility of (4, 1). If there is such a combination in short(code($bag(\omega)$)), then there is a combination $(4, 4', 1)$ in code($bag(\omega)$), but combination $(4', 1)$ is impossible there according to (LC-c).

Proof of (LMC-c). Due to (LC-a), the last term of C is $4'$. Due to (LC-c), the preceeding term of C is 3 or 4, and this term becomes the last term of short(C) when $4'$ is eliminated.

Proofs of (LMC-d) and (LMC-e). Follow from (31) and (32).

Lemma 6 is proved. Now we can estimate the sum in the right side of (28). Due to Lemma 6 we can represent this sum as

$$\sum_{C' \in LC_k} \text{chance}(C') = \sum_{C \in LMC_k} \sum_{C' \in \text{Long}(C)} \text{chance}(C'). \tag{33}$$

Let us estimate the right side. Due to the item (e), the result of the procedure (30) is not unique, generally speaking. However, the number of different possible outcomes, that is cardinality of Long(C), does not exceed 2^m, where m is the number of those terms of C, which equal 2. Also notice that $\text{chance}(C') = \text{chance}(C)$ whenever $C' \in \text{Long}(C)$ because $\text{chance}(1') = \text{chance}(4') = 1$. Thus for any $C \in LMC_k$

$$\sum_{C' \in \text{Long}(C)} \text{chance}(C') \leqslant 2^m \cdot \text{chance}(C) \leqslant \text{rate}(C),$$

where m has the same meaning. Substituting this into (33) we obtain

$$\sum_{C' \in LC_k} \text{chance}(C')) \leqslant \sum_{C \in LMC_k} \text{rate}(C). \tag{34}$$

It remains to prove this:

$$\sum_{k=1}^{\infty} \sum_{C \in \text{LMC}_k} \text{rate}(C) \leqslant \frac{300 \cdot \beta}{\alpha^2}. \tag{35}$$

Instead we shall prove this:

$$\sum_{C \in \mathrm{LMC}} \mathrm{Hshift}(C) \cdot \mathrm{rate}(C) \leqslant \frac{300 \cdot \beta}{\alpha^2}. \tag{36}$$

This is sufficient because the left sides of (35) and (36) are equal.

For any integer x and y, natural z and $k \in \{1, 2, 3, 4\}$ we denote by $S_k(x, y, z)$ the sum of rates of main codes satisfying conditions (LMC-a) and (LMC-b) of the definition of legal main codes, whose Hshift equals x, whose Vshift equals y and which have z terms, the last of which is k. It follows from the definition of $S_k(x, y, z)$ and conditions (LMC-c), (LMC-d), and (LMC-e) of (29) that

$$\sum_{C \in \mathrm{LMC}} \mathrm{Hshift}(C) \cdot \mathrm{rate}(C) \leqslant \sum_{x=1}^{\infty} \sum_{y=0}^{\infty} \sum_{z=1}^{\infty} x \cdot (S_3(x, y, z) + S_4(x, y, z)). \tag{37}$$

Due to condition (LMC-a) of (29), the numbers $S_k(x, y, z)$ satisfy the initial condition

$$S_k(x, y, 1) = \begin{cases} 1 & \text{if} \quad x = 0,\ y = -1, \text{ and } k = 1, \\ 0 & \text{in all the other cases} \end{cases}$$

and due to condition (LMC-b) they satisfy the transition equations

$$\begin{cases} S_1(x, y, z+1) = S_1(x, y+1, z) + S_2(x, y+1, z) + S_3(x, y+1, z), \\ S_2(x, y, z+1) = 2\beta \cdot (S_1(x-1, y, z) + S_2(x-1, y, z) \\ \qquad\qquad + S_3(x-1, y, z) + S_4(x-1, y, z)), \\ S_3(x, y, z+1) = \alpha \cdot (S_2(x+1, y-1, z) + S_3(x+1, y-1, z) \\ \qquad\qquad + S_4(x+1, y-1, z)), \\ S_4(x, y, z+1) = (1-\alpha) \cdot (S_2(x, y-1, z) + S_3(x, y-1, z) + S_4(x, y-1, z)). \end{cases}$$

To estimate (37), let us use sums

$$\begin{cases} S_1(z) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} p^{-x} q^{-y} S_1(x, y, z), \\ S_2(z) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} p^{-x} q^{-y} S_2(x, y, z), \\ S_3(z) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} p^{-x} q^{-y} S_3(x, y, z), \\ S_4(z) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} p^{-x} q^{-y} S_4(x, y, z), \end{cases} \tag{38}$$

where p, q are positive parameters, which we need to choose. The following values are sufficient to obtain our estimations:

$$p = 1/3 \qquad \text{and} \qquad q = 1 - \alpha/6. \tag{39}$$

However, it is convenient to keep using letters p and q for a while. Due to our choice of p and q and since $x < 3^x$ for all integer x, the sum (37) is estimated by

$$\sum_{z=1}^{\infty} (S_3(z) + S_4(z)), \tag{40}$$

so it remains to estimate the sum (40).

The quantities (38) satisfy the initial conditions

$$S_1(1) = \frac{1}{q}, \qquad S_2(1) = S_3(1) = S_4(1) = 0$$

and recurrence conditions

$$\begin{cases} S_1(z+1) = q(S_1(z) + S_2(z) + S_3(z)), \\ S_2(z+1) = 2\beta/p(S_1(z) + S_2(z) + S_3(z) + S_4(z)), \\ S_3(z+1) = p\alpha/q(S_2(z) + S_3(z) + S_4(z)), \\ S_4(z+1) = (1-\alpha)/q(S_2(z) + S_3(z) + S_4(z)). \end{cases}$$

Notice that $S_3(z)$ and $S_4(z)$ are proportional, namely for every z they relate as $p \cdot \alpha$ to $(1-\alpha)$, so we may go to other quantities

$$S_1^*(z) = S_1(z), \qquad S_2^*(z) = S_2(z), \qquad S_3^*(z) = S_3(z) + S_4(z)$$

with initial conditions

$$S_1^*(1) = \frac{1}{q}, \qquad S_2^*(1) = S_3^*(1) = 0 \tag{41}$$

and recurrence conditions

$$\begin{cases} S_1^*(z+1) = q(S_1^*(z) + S_2^*(z)) + p \cdot \alpha/r\, S_3^*(z), \\ S_2^*(z+1) = 2\beta/p(S_1^*(z) + S_2^*(z) + S_3^*(z)), \\ S_3^*(z+1) = r(S_2^*(z) + S_3^*(z)), \end{cases}$$

where we have denoted

$$r = \frac{(1-\alpha)+p\cdot\alpha}{q}. \tag{42}$$

Introducing a vector $S^*(z)=(S_1^*(z), S_2^*(z), S_3^*(z))$, we can write these recurrence conditions as $S^*(z+1)=S^*(z)\cdot M$, whence $S^*(z)=S^*(1)\cdot M^{z-1}$, where M is the matrix

$$M = \begin{pmatrix} q & 2\beta/p & 0 \\ q & 2\beta/p & r \\ p\cdot\alpha/r & 2\beta/p & r \end{pmatrix}.$$

Notice that in the spirit of our article we write matrices on the right side of vectors, so vectors are horizontal. Eigen-vectors of M are roots of the equation

$$|M-\lambda_{\max}\cdot E|=0$$

(where E is the identity matrix), which can be simplified to

$$2\beta\cdot(\lambda^2-(1-\alpha)) = p\lambda(\lambda-q)\,(\lambda-r). \tag{43}$$

Let us first consider the case $\beta=0$. In this case all the eigen-values of M can be written explicitly: they equal q, r and zero and it is easy to show that $q>r>0$ for all α, so q is the greatest eigen-value.

Now let $\beta>0$. Remember that $\beta\leqslant\alpha^2/300$. From Perron–Frobenius theorem, M has the "maximal" eigen-value $\lambda_{\max}$, which is real and positive and which is not less than absolute values of all the other eigen-values of M. If $\beta=0$, $\lambda_{\max}=q$ and it is strictly greater than all the other eigen-values (which are real and non-negative in this case, as we have seen). When β grows from zero to $\alpha^2/300$, $\lambda_{\max}$ also grows and still exceeds absolute values of all the other eigen-values.

All the components of the eigen-vector V corresponding to $\lambda_{\max}$ can be chosen real and non-negative. In the present case the first component of V is not zero, so we may assume that $V=(V_1, V_2, V_3)$ is normed in such a way that $V_1=1$. Then all the components of our initial vector (41) are not greater that the corresponding components of the vector V multiplied by $6/5$, because

$$S_1^*(1)=\frac{1}{q}\leqslant\frac{6}{5}V_1, \qquad S_2^*(1)=0\leqslant\frac{6}{5}V_2, \qquad S_3^*(1)=0\leqslant\frac{6}{5}V_3.$$

Hence and from non-negativity of all elements of M,

$$S_i(z) \leqslant \tfrac{6}{5} V_i \cdot \lambda_{\max}^z \qquad \text{for all } z \text{ and } i.$$

Therefore $S_3^*(z) \leqslant 6/5 \cdot V_3 \cdot \lambda_{\max}^z$, whence we can estimate the sum (37) as well as the sum (40) as follows:

$$\sum_{z=1}^{\infty} (S_3(z) + S_4(z)) = \sum_{z=1}^{\infty} S_3^*(z) \leqslant \frac{6}{5} \cdot V_3 \cdot \sum_{z=1}^{\infty} \lambda_{\max}^z = \frac{6}{5} \cdot \frac{V_3}{1-\lambda_{\max}}. \tag{44}$$

To estimate this expression, we need to estimate V_3 from above and $1-\lambda_{\max}$ from below. Let us first estimate $1-\lambda_{\max}$, for which we need to estimate $\lambda_{\max}$. From (43)

$$\frac{\lambda_{\max} - q}{2\beta} = \frac{\lambda_{\max}^2 - (1-\alpha)}{p\lambda_{\max}(\lambda_{\max} - r)}. \tag{45}$$

To estimate $\lambda_{\max}$ we need to estimate the left side of this expression. First we estimate the numerator of the right side:

$$\lambda_{\max}^2 - (1-\alpha) \leqslant 1 - (1-\alpha) = \alpha.$$

Now to estimate the denominator. Since $p = 1/3$ and $\lambda_{\max} \geqslant q = 1 - \alpha/6 \geqslant 5/6$,

$$p \cdot \lambda_{\max} \geqslant 5/18. \tag{46}$$

Also notice that $q - r \geqslant \alpha/3$, whence $\lambda_{\max} - r \geqslant q - r \geqslant \alpha/3$. So we can conclude that

$$p\lambda_{\max}(\lambda_{\max} - r) \geqslant \frac{5}{18} \cdot \frac{\alpha}{3} = \frac{5\alpha}{54}.$$

Now we can estimate the right side and therefore the left side of (45):

$$\frac{\lambda_{\max} - q}{2\beta} \leqslant \frac{\alpha}{5\alpha/54} = \frac{54}{5}.$$

Since $\beta \leqslant \alpha^2/300$,

$$\lambda_{\max} - q \leqslant \frac{2\alpha^2}{300} \cdot \frac{54}{5} = \frac{9\alpha}{125}.$$

Remember that $q = 1 - \alpha/6$. Therefore

$$1 - \lambda_{\max} = (1-q) - (\lambda_{\max} - q) \geqslant \frac{\alpha}{6} - \frac{9\alpha}{125} = \frac{71\alpha}{750}. \tag{47}$$

Thus the denominator of (44) is estimated. Now let us estimate the numerator, i.e., V_3, using its explicit representation:

$$V_3 = \frac{2\beta \cdot r}{p\lambda_{\max}(\lambda_{\max} - (2\beta/p + r))}. \tag{48}$$

It is easy to show that $r \leqslant 1 - \alpha/2$. Therefore the numerator of (48) does not exceed 2β. To estimate the denominator, remember that $\lambda_{\max} \geqslant q = 1 - \alpha/6$ and $2\beta/p = 6\beta \leqslant \alpha^2/50 \leqslant \alpha/50$. Therefore

$$2\beta/p + r \leqslant \frac{\alpha}{50} + 1 - \frac{\alpha}{2}.$$

Using (46), we estimate the denominator:

$$p\lambda_{\max}(\lambda_{\max} - (2\beta/p + r)) \geqslant \frac{5}{18} \cdot \left(1 - \frac{\alpha}{6} - \frac{\alpha}{50} - 1 + \frac{\alpha}{2}\right) \geqslant \frac{47\alpha}{540}.$$

Thus

$$V_3 \leqslant \frac{2\beta}{47\alpha/540} = \frac{1080\beta}{47\alpha}.$$

Hence and from (47),

$$\frac{6}{5} \times \frac{V_3}{1 - \lambda_{\max}} \leqslant \frac{6}{5} \times \frac{1080\beta}{47\alpha} \times \frac{750}{71\alpha} \leqslant \frac{300\beta}{\alpha^2}.$$

The inequality (35) is proved. Collecting together the equality (21), the inequalities (28) and (34) summed over k, and (35), we prove Theorem 1.

APPENDIX

The following figure illustrates our constructions.

This figure shows a possible (that is, having a positive probability) fragment of our process ν. The transformation from y to $y+1$ is done by Flip_β if y is even and by Ann_α if y is odd. The figure includes six instances of minuses turning into pluses due to the action of Flip_β (for the values

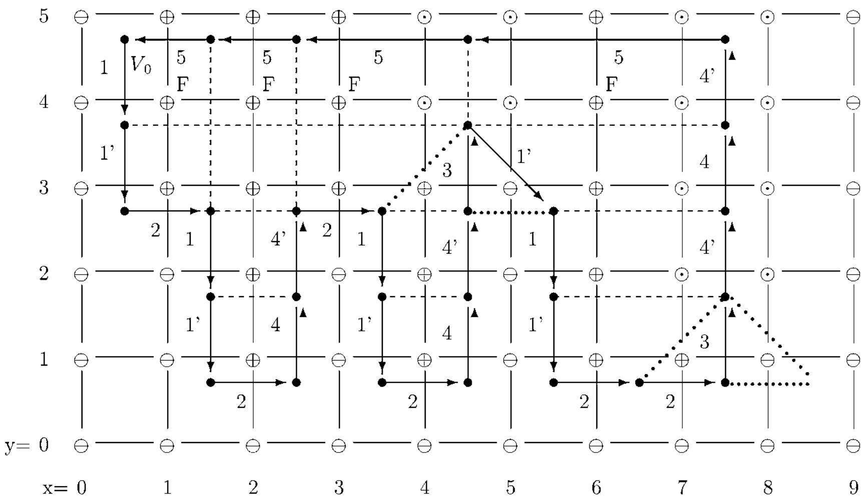

Figure 1.

1, 2, 3, 4, 6, 7 of x) and two instances of annihilation due to the action of Ann_α: the plus at (7, 1) annihilates with the minus at (8, 1) and the plus at (4, 3) annihilates with the minus at (5, 3). The leftmost and rightmost columns are filled with zeros since these zeros never were disturbed by our operators. For the leftmost column it displays the fact that our configuration belongs to Ω_0. The rightmost column with this property exists a.s. There are four flowers between these columns, namely (1, 4), (2, 4), (3, 4), (6, 4), marked with the letter F. The path $tour(\omega)$ surrounding the union of dual-roots is shown with thick vectors. The vertex V_0 is in its left upper corner. The vertices inside this path (all marked with pluses) are roots. Dual-roots, that is faces inside $tour(\omega)$, are separated from each other by dotted lines. To clarify the action of annihilation, boundaries of two triangles outside $tour(\omega)$ (dual of (8, 1) and (5, 3)) are shown by dotted lines also. Types of steps of $tour(\omega)$ are shown near each step. These types form the code of $tour(\omega)$, which is

$$11'211'244'211'244'31'11'2234'44'5555.$$

This code includes all the types except $2'$ and $2''$ and all possible combinations of any two of these types. (We especially chose a configuration with this property.) The code of $bag(\omega)$ is the same without fives and short(code($bag(\omega)$)) is 121242124312234.

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REFERENCES

1. A. Toom, N. Vasilyev, O. Stavskaya, L. Mityushin, G. Kurdyumov, and S. Pirogov, Discrete local Markov systems, in *Stochastic Cellular Systems: Ergodicity, Memory, Morphogenesis*, R. Dobrushin, V. Kryukov, and A. Toom, eds., Nonlinear Science: Theory and applications (Manchester University Press, 1990), pp. 1–182.
2. P. Ferreira de Lima and A. Toom, Dualities useful in bond percolation. Submitted to *Bulletin Brazilian Mathematical Society*.
3. T. M. Liggett, *Interacting Particle Systems* (Springer, 1985).
4. T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes* (Springer, 1999).
5. O. N. Stavskaya and I. I. Piatetski-Shapiro, On homogeneous nets of spontaneously active elements, *Systems Theory Res.* **20**:75–88 (1971).
6. A. Toom, A family of uniform nets of formal neurons, *Soviet Math. Doklady* **9**:1338–1341 (1968).
7. A. Toom, Cellular automata with errors: Problems for students of probability, *Topics in Contemporary Probability and Its Applications,* J. L. Snell, ed., *Series Probability and Stochastics*, R. Durrett and M. Pinsky, eds. (CRC Press, 1995), pp. 117–157.
8. A. Toom, Particle systems with variable length, *Bulletin of the Brazilian Mathematical Society* **33**(3):419–425 (November 2002).